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Regularity of large solutions for the compressible magnetohydrodynamic equations

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Full list of author information is available at the end of the article**Abstract**

In this paper, we consider the initial-boundary value problem of one-dimensional compressible magnetohydrodynamics flows. The existence and continuous dependence of global solutions in H^1 have been established in Chen and Wang (Z Angew Math Phys 54, 608-632, 2003). We will obtain the regularity of global solutions under certain assumptions on the initial data by deriving some new a priori estimates.

Keywords: magnetohydrodynamics (MHD), global solutions, regularity, initial-boundary value problem

1 Introduction

Magnetohydrodynamics (MHD) is concerned with the flow of electrically conducting fluids in the presence of magnetic fields, either externally applied or generated within the fluid by inductive action. The application of magnetohydrodynamics covers a very wide range of physical areas from liquid metals to cosmic plasmas, for example, the intensely heated and ionized fluids in an electromagnetic field in astrophysics, geophysics, high-speed aerodynamics, and plasma physics. There is a complex interaction between the magnetic and fluid dynamic phenomena, and both hydrodynamic and electrodynamic effects have to be considered. For convenience, we consider the following plane magnetohydrodynamic equations in the Lagrangian coordinate system:

$$v_t - u_\gamma = 0, \quad (1.1)$$

$$u_t + \left(p + \frac{1}{2} |b|^2\right)_\gamma = \left(\frac{\lambda u_\gamma}{v}\right)_\gamma, \quad (1.2)$$

$$w_t - b_\gamma = \left(\frac{\mu w_\gamma}{v}\right)_\gamma, \quad (1.3)$$

$$(vb)_t - w_\gamma = \left(\frac{vb_\gamma}{v}\right)_\gamma, \quad (1.4)$$

$$E_t + \left(u\left(p + \frac{1}{2} |b|^2\right) - w \cdot b\right)_\gamma = \left(\frac{\lambda u u_\gamma + \mu w \cdot w_\gamma + vb \cdot b_\gamma + \kappa \theta_\gamma}{v}\right)_\gamma. \quad (1.5)$$

Here, v , u , w , b , θ , and p are the specific volume, the longitudinal velocity, the transverse velocity, the transverse magnetic field, the absolute temperature, and the pressure, respectively; λ , μ , ν , and κ are the bulk viscosity coefficient, the shear viscosity coefficient, the magnetic diffusivity, and the heat conductivity, respectively.

We consider problem (1.1)-(1.5) in the region $\{y \in \Omega: = (0, 1), t \geq 0\}$ under the initial-boundary conditions

$$(v, u, w, b, \theta)|_{t=0} = (v_0, u_0, w_0, b_0, \theta)(y), \quad y \in \Omega, \quad (1.6)$$

$$(u, w, b, \theta_y)|_{\partial\Omega} = 0. \quad (1.7)$$

In this paper, we focus on an initial-boundary problem for the magnetohydrodynamic flows of a perfect gas with following equations of state:

$$p = \frac{R\theta}{v}, \quad e = c_v\theta,$$

where R is the gas constant and c_v is the heat capacity of the gas at constant volume. For concreteness, we assume that λ , μ , and ν are constants, and κ depends on the temperature θ with $C_1 \leq \kappa(\theta)/(1 + \theta^r) \leq C_2$ for some positive constants C_1 , C_2 and, $r \geq 2$. The growth condition assumed on κ is motivated by the physical fact: $\kappa \propto \theta^{5/2}$ for important physical regimes (see [1,2]). The total energy of the magnetohydrodynamics flows is

$$E = e + \frac{1}{2}(u^2 + |w|^2) + \frac{1}{2}\nu |b|^2.$$

Before showing our main results, let us first recall the related results in the literature. For the one-dimensional ideal gas, i.e.,

$$e = c_v\theta, \quad \sigma = \frac{R\theta}{v} + \mu \frac{u_y}{v}, \quad Q = -\kappa \frac{\theta_y}{v}, \quad (1.8)$$

with suitable positive constants c_v , R . Kazhikhov and Shelukhin [3-5], Kawashima and Nishida [6] established the existence of global smooth solutions. Zheng and Qin [7] proved the existence of maximal attractors in H^i ($i = 1, 2$). However, under very high temperatures and densities, constitutive relations (1.8) become inadequate. Thus, a more realistic model would be a linearly viscous gas (or Newtonian fluid)

$$\sigma(v, \theta, u_y) = -p(v, \theta) + \frac{\mu(v, \theta)}{v} u_y \quad (1.9)$$

satisfying Fourier's law of heat flux

$$Q(v, \theta, \theta_y) = -\frac{\kappa(v, \theta)}{v} \theta_y \quad (1.10)$$

whose internal energy e and pressure p are coupled by the standard thermodynamical relation (1.8). In this case, Kawohl [8] obtained the existence of global solutions with the exponents $r \in [0, 1]$, $q \geq 2r + 2$. Jiang [9] also established the global existence with basically same constitutive relations as those in [8] but with the exponents $r \in [0, 1]$, $q \geq r + 1$. When the exponents q , r satisfy the more general constitutive relations than those in [8,9], Qin [10] established the regularity and asymptotic behavior of

global solutions with arbitrary initial data for a one-dimensional viscous heat-conductive real gas.

For the radiative and reactive gas, Ducomet [11] established the global existence and exponential decay in H^1 of smooth solutions, and Umehara and Tani [12] proved the global existence of smooth solutions for a self-gravitating radiative and reactive gas.

For the radiative magnetohydrodynamic equations with self-gravitation, Ducomet and Feireisl [13] proved the existence of global-in-time solutions of this problem with arbitrarily large initial data and conservative boundary conditions on a bounded spatial domain in \mathbb{R}^3 . Recently, under the technical condition that $\kappa(\rho, \theta)$ satisfies

$$k_1(1 + \theta^q) \leq \kappa(\rho, \theta) \leq k_2(1 + \theta^q), \quad k_1(1 + \theta^q) \leq |\kappa_\rho(\rho, \theta)| \leq k_2(1 + \theta^q),$$

for some $q > \frac{5}{2}$, Zhang and Xie [14] investigated the existence of global smooth solutions.

For the non-radiative and non self-gravitation magnetohydrodynamic flows, there have been a number of studies under various conditions by several authors (see, e.g., [2,15-22]). The existence and uniqueness of local smooth solutions were first obtained in [21]; moreover, the existence of global smooth solutions with small smooth initial data was shown in [20]. Chen and Wang [15] investigated a free boundary problem with general large initial data with exponents $r \in [0, 1]$, $q \geq 2r + 2$. Under the technical condition that $\kappa(\rho, \theta)$ satisfies

$$C^{-1}(1 + \theta^q) \leq \kappa(\rho, \theta) \leq C(1 + \theta^q)$$

for $q \geq 2$, Chen and Wang [16] also proved the existence and continuous dependence of global strong solutions with large initial data. Wang [22] established large solutions to the initial-boundary value problem for planar magnetohydrodynamics. Under the technical condition upon

$$\kappa(\rho, \theta) \equiv \kappa(\rho) > \frac{C}{\rho},$$

Fan et al. [18] investigated the uniqueness of the weak solutions of MHD with Lebesgue initial data. Fan et al. [19] also considered a one-dimensional plane compressible MHD flows and proved that as the shear viscosity goes to zero, global weak solutions converge to a solution of the original equations with zero shear viscosity. The uniqueness and continuous dependence of weak solutions for the Cauchy problem have been proved by Hoff and Tsyganov [17].

As mentioned above, the global existence in $H_+^i (i = 2, 4)$ of global solutions has never been studied for Equations (1.1)-(1.5) of the nonlinear one-dimensional compressible magnetohydrodynamics flows with initial-boundary conditions (1.6)-(1.7). The main aim of this paper is to prove the regularity of solutions in the subspace H_+^i of $(H^i[0, 1])^7 (i = 2, 4)$ for systems (1.1)-(1.7). In order to obtain higher regularity of global solutions, there are many complicated estimates on higher derivations of solutions to be involved, this is our main difficulty. To overcome this difficulty, we should use some proper embedding theorems, the interpolation techniques as well as many delicate estimates. This is the novelty of the paper.

We define three spaces as follows:

$$\begin{aligned} H_+^1 &= \left\{ (v, u, w, b, \theta) \in (H^1(\Omega))^7 : v(x) > 0, \theta(x) > 0, \quad x \in \Omega, \right. \\ &\quad \left. u(0) = u(1) = 0, w(0) = w(1) = b(0) = b(1) = 0 \right\}, \\ H_+^i &= \left\{ (v, u, w, b, \theta) \in (H^i(\Omega))^7 : v(x) > 0, \theta(x) > 0, \quad x \in \Omega, \right. \\ &\quad \left. u(0) = u(1) = 0, w(0) = w(1) = b(0) = b(1) = 0, \right. \\ &\quad \left. \theta'(0) = \theta'(1) = 0 \right\}, \quad i = 2, 4. \end{aligned}$$

The notation in this paper will be stated as follows:

L^p , $1 \leq p \leq +\infty$, $W^{m,p}$, $m \in \mathbb{N}$, $H^1 = W^{1,2}$, $H_0^1 = W_0^{1,2}$ denote the usual (Sobolev) spaces on Ω . In addition, $\|\cdot\|_B$ denotes the norm in the space B , we also put $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$. Constants C_i ($i = 1, 2, 3, 4$) depend on the H_+^i norm of the initial data $(v_0, u_0, w_0, b_0, \theta_0)$ and $T > 0$.

Now we are in a position to state our main results.

Theorem 1.1 Assume that the initial data $(v_0, u_0, w_0, b_0, \theta_0) \in H_+^2$ and e, p , and κ are C^3 functions. Then, the problem (1.1)-(1.7) admits a unique global solution $(v(t), u(t), w(t), b(t), \theta(t)) \in H_+^2$ such that for any $T > 0$,

$$\begin{aligned} &\|v(t) - \bar{v}\|_{H^2}^2 + \|u(t)\|_{H^2}^2 + \|w(t)\|_{H^2}^2 + \|b(t)\|_{H^2}^2 + \|\theta(t) - \bar{\theta}\|_{H^2}^2 + \|u_t(t)\|^2 + \|w_t(t)\|^2 \\ &\quad + \|b_t(t)\|^2 + \|\theta_t(t)\|^2 + \int_0^t (\|v - \bar{v}\|_{H^2}^2 + \|u\|_{H^3}^2 + \|w\|_{H^3}^2 + \|b\|_{H^3}^2 + \|\theta - \bar{\theta}\|_{H^3}^2 \\ &\quad + \|u_{ty}\|^2 + \|w_{ty}\|^2 + \|b_{ty}\|^2 + \|\theta_{ty}\|^2)(s) ds \leq C_2, \quad \forall t \in [0, T], \end{aligned} \quad (1.11)$$

where $\bar{v} = \int_0^1 v dy = \int_0^1 v_0 dy$, constant $\bar{\theta} > 0$ is determined by

$$e(\bar{v}, \bar{\theta}) = \int_0^1 \left(\frac{1}{2} (u_0^2 + |w_0|^2 + v_0 |b_0|^2) + e(v_0, \theta_0) \right) (y) dy.$$

Theorem 1.2 Assume that the initial data $(v_0, u_0, w_0, b_0, \theta_0) \in H_+^4$ and e, p , and κ are C^5 functions on $0 < v < +\infty$ and $0 \leq \theta < +\infty$. Then, the problem (1.1)-(1.7) admits a unique global solution $(v(t), u(t), w(t), b(t), \theta(t)) \in H_+^4$ such that for any $T > 0$,

$$\begin{aligned} &\|v(t) - \bar{v}\|_{H^4}^2 + \|u(t)\|_{H^4}^2 + \|w(t)\|_{H^4}^2 + \|b(t)\|_{H^4}^2 + \|\theta(t) - \bar{\theta}\|_{H^4}^2 + \|u_{tt}(t)\|^2 + \|w_{tt}(t)\|^2 \\ &\quad + \|b_{tt}(t)\|^2 + \|u_t(t)\|_{H^2}^2 + \|w_t(t)\|_{H^2}^2 + \|b_t(t)\|_{H^2}^2 + \|\theta_t(t)\|_{H^2}^2 + \|\theta_{tt}(t)\|^2 \\ &\quad + \int_0^t (\|v - \bar{v}\|_{H^4}^2 + \|u\|_{H^5}^2 + \|w\|_{H^5}^2 + \|b\|_{H^5}^2 + \|\theta - \bar{\theta}\|_{H^5}^2 + \|u_t\|_{H^3}^2 + \|w_t\|_{H^3}^2 + \|b_t\|_{H^3}^2 \\ &\quad + \|\theta_t\|_{H^3}^2 + \|u_{tt}\|_{H^1}^2 + \|w_{tt}\|_{H^1}^2 + \|b_{tt}\|_{H^1}^2 + \|\theta_{tt}\|_{H^1}^2)(s) ds \leq C_4, \quad \forall t \in [0, T]. \end{aligned} \quad (1.12)$$

2 Proof of Theorem 1.1

In this section, we study the global existence of problem (1.1)-(1.7) in H_+^2 by establishing a series of priori estimates. Without loss of generality, we take $c_v = R = 1$. We begin with the following lemma.

Lemma 2.1 Assume that the initial data $(v_0, u_0, w_0, b_0, \theta_0) \in H_+^1$ and e, p , and κ are C^2 functions on $0 < v < +\infty$ and $0 \leq \theta < +\infty$ and there exists a positive constant C_0 such that

$$0 < C_0^{-1} \leq v_0(\gamma) \leq C_0, \quad 0 < C_0^{-1} \leq \theta_0(\gamma) \leq C_0.$$

Then, for the initial data $(v_0, u_0, w_0, b_0, \theta_0) \in H^1_+$ the problem (1.1) -(1.7) admits a unique global solution $(v(t), u(t), w(t), b(t), \theta(t)) \in H^1_+$ such that for any $T > 0$

$$0 < C_1^{-1} \leq v(\gamma, t) \leq C_1, \quad 0 < C_1^{-1} \leq \theta(\gamma, t) \leq C_1, \quad \forall (\gamma, t) \in [0, 1] \times [0, T] \quad (2.1)$$

and for any $t \in [0, T]$,

$$\begin{aligned} & \|v(t) - \bar{v}\|_{H^1}^2 + \|u(t)\|_{H^1}^2 + \|w(t)\|_{H^1}^2 + \|b(t)\|_{H^1}^2 + \|\theta(t) - \bar{\theta}\|_{H^1}^2 + \int_0^t (\|v - \bar{v}\|_{H^1}^2 + \|u\|_{H^2}^2 \\ & + \|w\|_{H^2}^2 + \|b\|_{H^2}^2 + \|\theta - \bar{\theta}\|_{H^2}^2 + \|u_t\|^2 + \|w_t\|^2 + \|b_t\|^2 + \|\theta_t\|^2) (s) ds \leq C_1. \end{aligned} \quad (2.2)$$

Proof. See, e.g., [16].

Lemma 2.2 Under the assumptions in Theorem 1.1, the following estimate holds:

$$\begin{aligned} & \|u_t(t)\|^2 + \|w_t(t)\|^2 + \|b_t(t)\|^2 + \|\theta_t(t)\|^2 \\ & + \int_0^t (\|u_{ty}\|^2 + \|w_{ty}\|^2 + \|b_{ty}\|^2 + \|\theta_{ty}\|^2) (s) ds \leq C_2, \quad \forall t \in [0, T]. \end{aligned} \quad (2.3)$$

Proof. Differentiating (1.2) with respect to t , multiplying the resultant by u_t , and then integrating the resulting equation over $Q_t := \Omega \times [0, t]$, we infer

$$\begin{aligned} & \|u_t(t)\|^2 + \int_0^t \|u_{ty}(s)\|^2 ds \\ & \leq \varepsilon \int_0^t \|u_{ty}(s)\|^2 ds + C_2 \int_0^t (\|\theta_t(s)\|^2 + \|\mathbf{b} \cdot \mathbf{b}_t(s)\|^2 + \|u_y(s)\|_{L^4}^4) ds \\ & \leq \varepsilon \int_0^t \|u_{ty}(s)\|^2 ds + C_2 \int_0^t (\|u_y(s)\|_{L^\infty}^2 + \|\theta_t(s)\|^2 + \|\mathbf{b}_t(s)\|^2) ds \\ & \leq C_2 + \varepsilon \int_0^t \|u_{ty}(s)\|^2 ds, \end{aligned}$$

which implies

$$\|u_t(t)\|^2 + \int_0^t \|u_{ty}(s)\|^2 ds \leq C_2. \quad (2.4)$$

Analogously, we have

$$\|w_t(t)\|^2 + \|b_t(t)\|^2 + \|\theta_t(t)\|^2 + \int_0^t (\|w_{ty}\|^2 + \|b_{ty}\|^2 + \|\theta_{ty}\|^2) (s) ds \leq C_2. \quad (2.5)$$

Thus, (2.3) follows from (2.4)-(2.5).

Lemma 2.3 *Under the assumptions in Theorem 1.1, the following estimate holds:*

$$\begin{aligned} & \|u_{\gamma\gamma}(t)\|^2 + \|w_{\gamma\gamma}(t)\|^2 + \|b_{\gamma\gamma}(t)\|^2 + \|\theta_{\gamma\gamma}(t)\|^2 + \int_0^t (\|u_{\gamma\gamma\gamma}\|^2 \\ & + \|w_{\gamma\gamma\gamma}\|^2 + \|b_{\gamma\gamma\gamma}\|^2 + \|\theta_{\gamma\gamma\gamma}\|^2)(s)ds \leq C_2, \quad \forall t \in [0, T]. \end{aligned} \quad (2.6)$$

Proof. Equation (1.2) can be rewritten as

$$u_t = -\frac{\lambda\theta_\gamma}{\nu} + \frac{\lambda\theta\nu_\gamma}{\nu} - \frac{\lambda u_\gamma \nu_\gamma}{\nu^2} + \frac{\lambda u_{\gamma\gamma}}{\nu} - \mathbf{b} \cdot \mathbf{b}_\gamma. \quad (2.7)$$

Using equation (2.7), Lemmas 2.1-2.2, Sobolev's embedding theorem and Young's inequality, we have

$$\begin{aligned} \|u_{\gamma\gamma}(t)\| & \leq C_2(\|u_t(t)\| + \|\theta_\gamma(t)\| + \|\mathbf{b} \cdot \mathbf{b}_\gamma(t)\| + \|\theta\nu_\gamma(t)\| + \|\nu_\gamma u_\gamma(t)\|) \\ & \leq C_2(\|u_t(t)\| + \|\theta_\gamma(t)\| + \|\theta(t)\|_{L^\infty} \|\nu_\gamma(t)\| + \|\mathbf{b}_\gamma(t)\|^2 + \|u_\gamma(t)\|_{L^\infty} \|\nu_\gamma(t)\|) \\ & \leq \varepsilon \|u_{\gamma\gamma}(t)\| + C_2(\|u_t(t)\| + 1), \end{aligned}$$

which leads to

$$\|u_{\gamma\gamma}(t)\| \leq C_2, \quad \int_0^t \|u_{\gamma\gamma\gamma}(s)\|^2 ds \leq C_2 \int_0^t \|u_{t\gamma}(s)\|^2 ds \leq C_2. \quad (2.8)$$

Similarly, we derive

$$\|w_{\gamma\gamma}(t)\| + \|b_{\gamma\gamma}(t)\| + \|\theta_{\gamma\gamma}(t)\| \leq C_2(\|w_t(t)\| + \|\mathbf{b}_t(t)\| + \|\theta_t(t)\| + 1) \leq C_2, \quad (2.9)$$

$$\int_0^t (\|w_{\gamma\gamma\gamma}\|^2 + \|b_{\gamma\gamma\gamma}\|^2 + \|\theta_{\gamma\gamma\gamma}\|^2)(s)ds \leq C_2. \quad (2.10)$$

Thus, (2.6) follows from (2.8)-(2.10).

Lemma 2.4 *Under the assumptions in Theorem 1.1, the following estimate holds:*

$$\|v_{\gamma\gamma}(t)\|^2 + \int_0^t \|v_{\gamma\gamma}(s)\|^2 ds \leq C_2, \quad \forall t \in [0, T]. \quad (2.11)$$

Proof. Differentiating (1.2) with respect to y , we obtain

$$\lambda \frac{d}{dt} \left(\frac{v_{\gamma\gamma}}{\nu} \right) + \frac{\theta}{\nu^2} v_{\gamma\gamma} = u_{t\gamma} + E(\gamma, t), \quad (2.12)$$

where

$$E(\gamma, t) = \frac{\theta_{\gamma\gamma}}{\nu} + \frac{2v_\gamma(\lambda u_{\gamma\gamma} - \theta_\gamma)}{\nu^2} + \frac{2v_\gamma^2(\theta - \lambda u_\gamma)}{\nu^3} + \mathbf{b} \cdot \mathbf{b}_{\gamma\gamma} + |\mathbf{b}_\gamma|^2.$$

Multiplying (2.12) by $\frac{v_{\gamma\gamma}}{\nu}$, integrating the resulting equation over Q_ν and then using the Young inequality and interpolation theorem, we can conclude

$$\begin{aligned} & \left| \frac{v_{\gamma\gamma}}{v}(t) \right|^2 + C_1^{-1} \int_0^t \left| \frac{v_{\gamma\gamma}}{v}(s) \right|^2 ds \leq \frac{1}{4C_1} \int_0^t \left| \frac{v_{\gamma\gamma}}{v}(s) \right|^2 ds + C_2 \int_0^t (\|u_{t\gamma}\|^2 + \|\theta_{\gamma\gamma}\|^2 \\ & \quad + \|v_\gamma u_{\gamma\gamma}\|^2 + \|v_\gamma\|_{L^4}^4 + \|u_\gamma v_\gamma^2\|^2 + \|b_\gamma\|_{L^4}^4 + \|b\|_{L^\infty}^2 \|b_{\gamma\gamma}\|^2) (s) ds \\ & \leq \frac{1}{2C_1} \int_0^t \left| \frac{v_{\gamma\gamma}}{v}(s) \right|^2 ds + C_2 \int_0^t (\|v_\gamma\|^2 + \|u_{t\gamma}\|^2 + \|u_{\gamma\gamma}\|_{L^\infty}^2 + \|\theta_{\gamma\gamma}\|^2 + \|b_{\gamma\gamma}\|^2) (s) ds, \end{aligned}$$

which, together with Lemmas 2.1-2.3, yields (2.11).

Proof of Theorem 1.1. By Lemmas 2.1-2.4, we complete the proof of Theorem 1.1.

3 Proof of Theorem 1.2

In this section, we study the global existence of problem (1.1)-(1.7) in H_+^4 by establishing a series of priori estimates. We begin with the following lemmas.

Lemma 3.1 *Under the assumptions in Theorem 1.2, the following estimates hold:*

$$\|u_{t\gamma}(\gamma, 0)\| + \|w_{t\gamma}(\gamma, 0)\| + \|w_{t\gamma}(\gamma, 0)\| + \|\theta_{t\gamma}(\gamma, 0)\| \leq C_3, \quad (3.1)$$

$$\begin{aligned} & \|u_{tt}(\gamma, 0)\| + \|w_{tt}(\gamma, 0)\| + \|b_{tt}(\gamma, 0)\| + \|\theta_{tt}(\gamma, 0)\| \\ & + \|u_{t\gamma\gamma}(\gamma, 0)\| + \|w_{t\gamma\gamma}(\gamma, 0)\| + \|b_{t\gamma\gamma}(\gamma, 0)\| + \|\theta_{t\gamma\gamma}(\gamma, 0)\| \leq C_3. \end{aligned} \quad (3.2)$$

Proof. We easily infer from (1.2), Lemma 2.1 and Theorems 1.1 that

$$\begin{aligned} \|u_t(t)\| & \leq C_3(\|v_\gamma(t)\| + \|\theta_\gamma(t)\| + \|u_{\gamma\gamma}(t)\| + \|u_\gamma(t)\|_{L^\infty} \|v_\gamma(t)\| + \|b(t)\|_{L^\infty} \|b_\gamma(t)\|) \\ & \leq C_3(\|v_\gamma(t)\| + \|\theta_\gamma(t)\| + \|u_{\gamma\gamma}(t)\| + \|b_\gamma(t)\|). \end{aligned}$$

Differentiating (1.2) with respect to γ , and using Theorem 1.1, we get

$$\|u_{t\gamma}(t)\| \leq C_3(\|v_\gamma(t)\|_{H^1} + \|\theta_\gamma(t)\|_{H^1} + \|u_\gamma(t)\|_{H^2} + \|b_\gamma(t)\|_{H^1}), \quad (3.3)$$

or

$$\|u_{\gamma\gamma\gamma}(t)\| \leq C_3(\|v_\gamma(t)\|_{H^1} + \|\theta_\gamma(t)\|_{H^1} + \|b_\gamma(t)\|_{H^1} + \|u_{t\gamma}(t)\|). \quad (3.4)$$

Differentiating (1.2) with respect to γ twice, using the embedding theorem and Theorem 1.1, we conclude

$$\|u_{t\gamma\gamma}(t)\| \leq C_3(\|v_\gamma(t)\|_{H^2} + \|\theta_\gamma(t)\|_{H^2} + \|u_\gamma(t)\|_{H^3} + \|b_\gamma(t)\|_{H^2}), \quad (3.5)$$

or

$$\|u_{\gamma\gamma\gamma\gamma}(t)\| \leq C_3(\|v_\gamma(t)\|_{H^2} + \|\theta_\gamma(t)\|_{H^2} + \|b_\gamma(t)\|_{H^2} + \|u_{t\gamma\gamma}(t)\|). \quad (3.6)$$

Similarly, we have

$$\begin{aligned} \|w_t(t)\| & \leq C_3(\|w_\gamma(t)\|_{H^1} + \|b_\gamma(t)\| + \|v_\gamma(t)\|), \\ \|w_{t\gamma}(t)\| & \leq C_3(\|w_\gamma(t)\|_{H^2} + \|b_\gamma(t)\|_{H^1} + \|v_\gamma(t)\|_{H^1}), \end{aligned} \quad (3.7)$$

or

$$\|w_{\gamma\gamma\gamma}(t)\| \leq C_3(\|b_\gamma(t)\|_{H^1} + \|v_\gamma(t)\|_{H^1} + \|b_{t\gamma}(t)\|), \quad (3.8)$$

$$\|w_{t\gamma\gamma}(t)\| \leq C_3(\|w_\gamma(t)\|_{H^3} + \|b_\gamma(t)\|_{H^2} + \|v_\gamma(t)\|_{H^2}), \quad (3.9)$$

or

$$\|w_{\gamma\gamma\gamma}(t)\| \leq C_3(\|b_\gamma(t)\|_{H^2} + \|v_\gamma(t)\|_{H^2} + \|w_{\eta\gamma}(t)\|), \quad (3.10)$$

$$\begin{aligned} \|b_t\| &\leq C_3(\|b_\gamma\|_{H^1} + \|w_\gamma\| + \|v_\gamma\|), \\ \|b_{\eta\gamma}(t)\| &\leq C_3(\|b_\gamma(t)\|_{H^2} + \|w_\gamma(t)\|_{H^1} + \|v_\gamma(t)\|_{H^1}), \end{aligned} \quad (3.11)$$

or

$$\|b_{\gamma\gamma}(t)\| \leq C_3(\|w_\gamma(t)\|_{H^1} + \|v_\gamma(t)\|_{H^1} + \|b_{\eta\gamma}(t)\|), \quad (3.12)$$

$$\|b_{\eta\gamma}(t)\| \leq C_3(\|b_\gamma(t)\|_{H^3} + \|w_\gamma(t)\|_{H^2} + \|v_\gamma(t)\|_{H^2}), \quad (3.13)$$

or

$$\|b_{\gamma\gamma\gamma}(t)\| \leq C_3(\|w_\gamma(t)\|_{H^2} + \|v_\gamma(t)\|_{H^2} + \|b_{\eta\gamma}(t)\|), \quad (3.14)$$

$$\begin{aligned} \|\theta_t(t)\| &\leq C_3(\|u_\gamma(t)\| + \|v_\gamma(t)\| + \|\theta_{\eta\gamma}(t)\| + \|u_\gamma(t)\|_{L^\infty} \|u_\gamma(t)\| \\ &\quad + \|w_\gamma(t)\|_{L^\infty} \|w_\gamma(t)\| + \|b_\gamma(t)\|_{L^\infty} \|b_\gamma(t)\| + \|\theta_\gamma(t)\|_{L^\infty} \|\theta_\gamma(t)\|), \\ &\leq C_3(\|\theta_{\eta\gamma}(t)\| + \|u_{\eta\gamma}(t)\| + \|w_{\eta\gamma}(t)\| + \|b_{\eta\gamma}(t)\|), \end{aligned} \quad (3.15)$$

$$\begin{aligned} \|\theta_{\eta\gamma}(t)\| &\leq C_3(\|\theta_t(t)\| + \|\theta_\gamma(t)\|_{H^2} + \|v_\gamma(t)\|_{H^1} + \|u_\gamma(t)\|_{H^1} + \|w_\gamma(t)\|_{H^1} \\ &\quad + \|b_\gamma(t)\|_{H^1}), \end{aligned} \quad (3.16)$$

or

$$\begin{aligned} \|\theta_{\gamma\gamma}(t)\| &\leq C_3(\|v_\gamma(t)\|_{H^1} + \|u_\gamma(t)\|_{H^1} + \|w_\gamma(t)\|_{H^1} + \|b_\gamma(t)\|_{H^1} \\ &\quad + \|\theta_{\eta\gamma}(t)\|), \end{aligned} \quad (3.17)$$

$$\|\theta_{\eta\gamma}(t)\| \leq C_3(\|\theta_\gamma(t)\|_{H^3} + \|v_\gamma(t)\|_{H^2} + \|u_\gamma(t)\|_{H^2} + \|w_\gamma(t)\|_{H^2} + \|b_\gamma(t)\|_{H^2}), \quad (3.18)$$

or

$$\begin{aligned} \|\theta_{\gamma\gamma\gamma}(t)\| &\leq C_3(\|v_\gamma(t)\|_{H^2} + \|u_\gamma(t)\|_{H^2} + \|w_\gamma(t)\|_{H^2} + \|b_\gamma(t)\|_{H^2} \\ &\quad + \|\theta_{\eta\gamma}(t)\|). \end{aligned} \quad (3.19)$$

Differentiating (1.2) with respect to t , and using Theorem 1.1, (3.3), (3.5), (3.11)-(3.12) and (3.16), we derive

$$\|u_{tt}(t)\| \leq C_3(\|v_\gamma(t)\|_{H^2} + \|u_\gamma(t)\|_{H^3} + \|b_\gamma(t)\|_{H^2} + \|\theta_\gamma(t)\|_{H^2}). \quad (3.20)$$

Similarly, we can conclude

$$\|w_{tt}(t)\| \leq C_3(\|v_\gamma(t)\|_{H^2} + \|b_\gamma(t)\|_{H^2} + \|w_\gamma(t)\|_{H^3}), \quad (3.21)$$

$$\|b_{tt}(t)\| \leq C_3(\|v_\gamma(t)\|_{H^2} + \|b_\gamma(t)\|_{H^3} + \|w_\gamma(t)\|_{H^2}), \quad (3.22)$$

$$\|\theta_{tt}(t)\| \leq C_3(\|v_\gamma(t)\|_{H^2} + \|u_\gamma(t)\|_{H^2} + \|b_\gamma(t)\|_{H^2} + \|w_\gamma(t)\|_{H^2} + \|\theta_\gamma(t)\|_{H^3}). \quad (3.23)$$

Thus, (3.1) follows from (3.3), (3.7), (3.11) and (3.16), and (3.2) from (3.5), (3.9), (3.13), (3.18) and (3.20)-(3.23).

Lemma 3.2 *Under the assumptions in Theorem 1.2, the following estimates hold, for any $t \in [0, T]$,*

$$\|u_{tt}(t)\|^2 + \int_0^t \|u_{t\gamma}(s)\|^2 ds \leq C_3 + C_3 \int_0^t (\|b_{\eta\gamma}\|^2 + \|\theta_{\eta\gamma}\|^2)(s) ds, \quad (3.24)$$

$$\begin{aligned} & \|w_{tt}(t)\|^2 + \|b_{tt}(t)\|^2 + \int_0^t (\|w_{t\gamma}(s)\|^2 + \|b_{t\gamma}(s)\|^2) ds \\ & \leq C_3 + C_3 \int_0^t (\|b_{\eta\gamma}(s)\|^2 + \|w_{\eta\gamma}(s)\|^2) ds, \end{aligned} \quad (3.25)$$

$$\begin{aligned} & \|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{t\gamma}(s)\|^2 ds \leq C_3 + C_2 \varepsilon^{-1} \int_0^t \|\theta_{\eta\gamma}(s)\|^2 ds \\ & + C_1 \varepsilon \int_0^t (\|u_{\eta\gamma}\|^2 + \|u_{t\gamma}\|^2 + \|w_{\eta\gamma}\|^2 + \|w_{t\gamma}\|^2 + \|b_{\eta\gamma}\|^2 + \|b_{t\gamma}\|^2)(s) ds. \end{aligned} \quad (3.26)$$

Proof. Differentiating (1.2) with respect to t twice, multiplying the resulting equation by u_{tt} , performing an integration by parts, and using Lemma 2.1, we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_0^1 u_{tt}^2(\gamma, t) d\gamma \leq -\lambda \|u_{t\gamma}(t)\|^2 + C_4 (\|\theta_{tt}(t)\| + \|u_{\eta\gamma}(t)\| + \|\theta_t u_\gamma(t)\| \\ & \quad + \|b \cdot b_{tt}(t)\| + \|b_t^2(t)\|^2 + \|u_{\eta\gamma}(t)\| \|u_{t\gamma}(t)\|) \\ & \leq -C_1^{-1} \|u_{t\gamma}(t)\|^2 + C_4 (\|\theta_{tt}(t)\|^2 + \|b_{tt}(t)\|^2 + \|u_{\eta\gamma}(t)\|^2 + \|u_\gamma(t)\|^2 + \|\theta_t(t)\|^2). \end{aligned} \quad (3.27)$$

Thus, using Theorem 1.1 and Lemma 3.1, we get

$$\|u_{tt}(t)\|^2 + \int_0^t \|u_{t\gamma}(s)\|^2 ds \leq C_3 + C_3 \int_0^t (\|b_{\eta\gamma}\|^2 + \|\theta_{\eta\gamma}\|^2)(s) ds.$$

Analogously, we obtain

$$\begin{aligned} & \|w_{tt}(t)\|^2 + \|b_{tt}(t)\|^2 + \int_0^t (\|w_{t\gamma}\|^2 + \|b_{t\gamma}\|^2)(s) ds \\ & \leq C_3 + C_3 \int_0^t (\|w_{\eta\gamma}\|^2 + \|b_{\eta\gamma}\|^2)(s) ds. \end{aligned}$$

Equation (1.5) can be rewritten as

$$(c_v \theta)_t + p u_\gamma = \left(\frac{\kappa \theta_\gamma}{v} \right)_\gamma + \frac{\lambda u_\gamma^2 + \mu |w_\gamma|^2 + v |b_\gamma|^2}{v}. \quad (3.28)$$

Differentiating (3.28) with respect to t twice, multiplying the resulting equation by θ_{tt} in $L^2[0, 1]$ and integrating by parts, we arrive at

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \int_0^1 c_v \theta_{tt}(\gamma, t) d\gamma = - \int_0^1 \left(\frac{\kappa \theta_\gamma}{v} \right)_{tt} \theta_{t\gamma}(\gamma, t) d\gamma \\
& - \int_0^1 \left(p - \frac{\lambda u_\gamma}{v} \right) u_{t\gamma} \theta_{tt}(\gamma, t) d\gamma + \int_0^1 \left(\frac{\mu w_\gamma}{v} w_{t\gamma} + \frac{v b_\gamma}{v} b_{t\gamma} \right) \theta_{tt}(\gamma, t) d\gamma \\
& - 2 \int_0^1 \left(p_t - \left(\frac{\lambda u_\gamma}{v} \right)_t \right) u_{t\gamma} \theta_{tt}(\gamma, t) d\gamma + 2 \int_0^1 \left[\left(\frac{\mu w_\gamma}{v} \right)_t w_{t\gamma} + \left(\frac{v b_\gamma}{v} \right)_t b_{t\gamma} \right] \theta_{tt}(\gamma, t) d\gamma \\
& + \int_0^1 \left(-p + \frac{\lambda u_\gamma}{v} \right)_{tt} u_\gamma \theta_{tt}(\gamma, t) d\gamma + \int_0^1 \left[\left(\frac{\mu w_\gamma}{v} \right)_{tt} w_\gamma + \left(\frac{v b_\gamma}{v} \right)_{tt} b_\gamma \right] \theta_{tt}(\gamma, t) d\gamma \\
& = B_1 + B_2 + B_3 + B_4 + B_5 + B_6 + B_7.
\end{aligned}$$

By virtue of Theorem 1.1 and Lemmas 3.1-3.2, using the embedding theorem, we deduce for any $\varepsilon \in (0, 1)$,

$$\begin{aligned}
B_1 & \leq -C^{-1} \|\theta_{t\gamma}(t)\|^2 + C_2 \|\theta_{t\gamma}(t)\|_{L^\infty} (\|u_\gamma(t)\|^2 + \|\theta_t(t)\|^2) \|\theta_{t\gamma}(t)\| \\
& \quad + C_2 \left\| \left(\frac{\kappa}{v} \right)_{tt}(t) \right\|^2 \|\theta_\gamma(t)\|_{L^\infty} \|\theta_{t\gamma}(t)\| \\
& \leq -2C^{-1} \|\theta_{t\gamma}(t)\|^2 + C_2 (\|u_\gamma(t)\|^2 + \|\theta_t(t)\|^2 + \|u_{t\gamma}(t)\|^2 \\
& \quad + \|\theta_{t\gamma}(t)\|^2 + \|\theta_{tt}(t)\|^2 + \|\theta_{t\gamma\gamma}(t)\|^2), \\
B_2 & \leq \varepsilon \|u_{t\gamma}(t)\|^2 + C_2 \varepsilon^{-1} \|\theta_{tt}(t)\|^2, \\
B_3 & \leq \varepsilon (\|w_{t\gamma}(t)\|^2 + \|b_{t\gamma}(t)\|^2) + C_2 \varepsilon^{-1} \|\theta_{tt}(t)\|^2
\end{aligned}$$

and

$$\begin{aligned}
B_4 & \leq C_1 \int_0^1 (|\theta_t| + |u_\gamma| + |u_{t\gamma}| + |u_\gamma|^2) |\theta_{tt}| |u_{t\gamma}|(\gamma, t) d\gamma \\
& \leq C_2 \|\theta_{t\gamma}(t)\|^{\frac{1}{2}} \|u_{t\gamma\gamma}(t)\|^{\frac{1}{2}} (\|\theta_t(t)\| + \|u_\gamma(t)\| + \|u_{t\gamma}(t)\|) \|\theta_{tt}(t)\|
\end{aligned}$$

which implies

$$\begin{aligned}
\int_0^t B_4 ds & \leq C_2 \sup_{0 \leq s \leq t} \|\theta_{tt}(s)\| \left(\int_0^t \|u_{t\gamma}(s)\|^2 ds \right)^{\frac{1}{4}} \left(\int_0^t \|u_{t\gamma\gamma}(s)\|^2 ds \right)^{\frac{1}{4}} \\
& \quad \times \left(\int_0^t (\|u_{t\gamma}\|^2 + \|\theta_t\|^2 + \|u_\gamma\|^2)(s) ds \right)^{\frac{1}{2}} \\
& \leq \varepsilon \left(\sup_{0 \leq s \leq t} \|\theta_{tt}(s)\|^2 + \int_0^t \|u_{t\gamma\gamma}(s)\|^2 ds \right) + C_3 \varepsilon^{-3}.
\end{aligned}$$

$$\begin{aligned}
B_5 & \leq C_1 \int_0^1 [(|w_\gamma|^2 + |w_{t\gamma}|) |w_\gamma| + (|w_\gamma|^2 + |w_{t\gamma}|) |w_{t\gamma}|] |\theta_{tt}|(\gamma, t) d\gamma \\
& \leq C_2 \|w_{t\gamma}(t)\|_{L^\infty} (\|w_\gamma(t)\|^2 + \|w_{t\gamma}(t)\|) \|\theta_{tt}(t)\| + C_2 \|w_{t\gamma}(t)\|_{L^\infty} (\|b_\gamma(t)\|^2 + \|b_{t\gamma}(t)\|) \|\theta_{tt}(t)\|
\end{aligned}$$

which implies

$$\begin{aligned} \int_0^t B_5 ds &\leq \varepsilon \left(\sup_{0 \leq s \leq t} \|\theta_{tt}(s)\|^2 + \int_0^t (\|w_{\eta\gamma}\|^2 + \|w_{\eta\gamma}\|^2(s) ds) \right) + C_3 \varepsilon^{-3}, \\ B_6 &\leq C_2 \|u_\gamma(t)\|_{L^\infty} \|\theta_{tt}(t)\| \left[(\|\theta_t(t)\|_{L^\infty} + \|u_\gamma(t)\|_{L^\infty}) (\|\theta_t(t)\| + \|u_\gamma(t)\|) + \|\theta_{tt}(t)\| \right. \\ &\quad \left. + \|u_{\eta\gamma}(t)\| + \|u_\gamma(t)\| + \|u_{\eta\gamma}(t)\| \right] \\ &\leq C_2 \|\theta_{tt}(t)\| (\|\theta_t(t)\| + \|u_\gamma(t)\|_{H^1} + \|\theta_{\eta\gamma}(t)\| + \|\theta_{tt}(t)\| + \|u_{\eta\gamma}(t)\| + \|u_{\eta\gamma}(t)\|) \\ &\leq \varepsilon \|u_{\eta\gamma}(t)\|^2 + C_2 \varepsilon^{-1} (\|\theta_t(t)\|^2 + \|u_\gamma(t)\|_{H^1}^2 + \|\theta_{\eta\gamma}(t)\|^2 + \|\theta_{tt}(t)\|^2 + \|u_{\eta\gamma}(t)\|^2), \\ B_7 &\leq C_2 \|w_\gamma(t)\|_{L^\infty} \|\theta_{tt}(t)\| (\|w_{\eta\gamma}(t)\| + \|w_{\eta\gamma}(t)\| + \|w_\gamma(t)\|) + C_2 \|b_\gamma(t)\|_{L^\infty} \|\theta_{tt}(t)\| \\ &\quad \times (\|b_{\eta\gamma}(t)\| + \|b_{\eta\gamma}(t)\| + \|b_\gamma(t)\|) \\ &\leq C_2 \|\theta_{tt}(t)\| (\|w_{\eta\gamma}(t)\| + \|w_\gamma(t)\|_{H^1} + \|w_{\eta\gamma}(t)\| + \|b_{\eta\gamma}(t)\| + \|b_\gamma(t)\|_{H^1} + \|b_{\eta\gamma}(t)\|) \\ &\leq \varepsilon (\|w_{\eta\gamma}(t)\|^2 + \|b_{\eta\gamma}(t)\|^2) + C_2 \varepsilon^{-1} (\|\theta_{tt}(t)\|^2 + \|w_\gamma(t)\|_{H^1}^2 \\ &\quad + \|w_{\eta\gamma}(t)\|^2 + \|b_\gamma(t)\|_{H^1}^2 + \|b_{\eta\gamma}(t)\|^2). \end{aligned}$$

Thus, for $\varepsilon \in (0, 1)$ small enough, we derive from above estimates

$$\begin{aligned} \|\theta_{tt}(t)\|^2 + \int_0^t \|\theta_{\eta\gamma}(s)\|^2 ds &\leq C_2 \varepsilon^{-1} \int_0^t (\|\theta_{\eta\gamma}(s)\|^2 + \|\theta_{tt}(s)\|^2) ds + C_3 \varepsilon^{-3} \\ &\quad + C_1 \varepsilon \left[\sup_{0 \leq s \leq t} \|\theta_{tt}(s)\|^2 + \int_0^t (\|u_{\eta\gamma}\|^2 + \|u_{\eta\gamma}\|^2 + \|w_{\eta\gamma}\|^2 \right. \\ &\quad \left. + \|w_{\eta\gamma}\|^2 + \|b_{\eta\gamma}\|^2 + \|b_{\eta\gamma}\|^2(s) ds) \right]. \end{aligned} \quad (3.29)$$

Thus, taking supremum in t on the left-hand side of (3.29), picking $\varepsilon \in (0, 1)$ small enough, and using (3.23), we can derive estimate (3.26).

Lemma 3.3 *Under the assumptions in Theorem 1.2, the following estimates hold, for any $t \in [0, T]$,*

$$\begin{aligned} \|u_{\eta\gamma}(t)\|^2 + \int_0^t \|u_{\eta\gamma}(s)\|^2 ds \\ \leq C_3 \varepsilon^{-6} + C_2 \varepsilon^2 \int_0^t (\|b_{\eta\gamma}\|^2 + \|\theta_{\eta\gamma}\|^2 + \|u_{\eta\gamma}\|^2)(s) ds, \end{aligned} \quad (3.30)$$

$$\begin{aligned} \|w_{\eta\gamma}(t)\|^2 + \|b_{\eta\gamma}(t)\|^2 + \int_0^t (\|w_{\eta\gamma}\|^2 + \|b_{\eta\gamma}\|^2)(s) ds \\ \leq C_3 \varepsilon^{-6} + C_2 \varepsilon^2 \int_0^t (\|w_{\eta\gamma}\|^2 + \|b_{\eta\gamma}\|^2)(s) ds, \end{aligned} \quad (3.31)$$

$$\begin{aligned} \|\theta_{\eta\gamma}(t)\|^2 + \int_0^t \|\theta_{\eta\gamma}(s)\|^2 ds &\leq C_3 \varepsilon^{-6} \\ &\quad + C_2 \varepsilon^2 \int_0^t (\|b_{\eta\gamma}\|^2 + \|u_{\eta\gamma}\|^2 + \|w_{\eta\gamma}\|^2 + \|\theta_{\eta\gamma}\|^2 + \|\theta_{\eta\gamma}\|^2 \|\theta_{\eta\gamma}\|^2)(s) ds. \end{aligned} \quad (3.32)$$

Proof. Differentiating (1.2) with respect to y and t , multiplying the resulting equation by u_{ty} , and integrating by parts, we arrive at

$$\frac{1}{2} \frac{d}{dt} \|u_{ty}(t)\|^2 = D_0(y, t) + D_1(t), \quad (3.33)$$

where

$$D_0(y, t) = \sigma_{ty} u_{ty}|_{y=0}^{y=1}, \quad D_1(t) = - \int_0^1 \sigma_{ty} u_{t\eta\eta} dy, \quad \sigma = - \left(p + \frac{1}{2} |b|^2 - \frac{\lambda u_y}{v} \right).$$

We use Theorem 1.1, Lemma 2.1, the interpolation inequality and Poincaré's inequality to obtain

$$\begin{aligned} D_0 &\leq C_1 [(\|u_y(t)\|_{L^\infty} + \|\theta_t(t)\|_{L^\infty})(\|v_y(t)\|_{L^\infty} + \|\theta_y(t)\|_{L^\infty}) \\ &\quad + \|\mathbf{b}_t(t)\|_{L^\infty} \|\mathbf{b}_y(t)\|_{L^\infty} + \|\mathbf{b}_{ty}(t)\|_{L^\infty} \|\mathbf{b}(t)\|_{L^\infty} + \|\theta_{ty}(t)\|_{L^\infty} + \|u_y(t)\|_{L^\infty}^2 \\ &\quad + \|u_y(t)\|_{L^\infty} \|u_{\eta\eta}(t)\|_{L^\infty} + \|u_{ty}(t)\|_{L^\infty} \|v_y(t)\|_{L^\infty} + \|u_{\eta\eta}(t)\|_{L^\infty}] \|u_{ty}(t)\|_{L^\infty} \\ &\leq C_3(D_{01} + D_{01}) \|u_{ty}(t)\|^{\frac{1}{2}} \|u_{\eta\eta}(t)\|^{\frac{1}{2}}, \end{aligned} \quad (3.34)$$

where

$$\begin{aligned} D_{01} &= \|u_y(t)\|_{H^2} + \|\theta_t(t)\| + \|\theta_{ty}(t)\| + \|\mathbf{b}_t(t)\| + \|\mathbf{b}_{ty}(t)\|, \\ D_{02} &= \|\theta_{ty}(t)\|^{\frac{1}{2}} \|\theta_{\eta\eta}(t)\|^{\frac{1}{2}} + \|u_{\eta\eta}(t)\|^{\frac{1}{2}} \|u_{\eta\eta\eta}(t)\|^{\frac{1}{2}} + \|u_{\eta\eta}(t)\| \\ &\quad + \|u_{ty}(t)\|^{\frac{1}{2}} \|u_{\eta\eta}(t)\|^{\frac{1}{2}} + \|\mathbf{b}_{ty}(t)\|^{\frac{1}{2}} \|\mathbf{b}_{\eta\eta}(t)\|^{\frac{1}{2}}. \end{aligned}$$

Using the Young inequality several times, we derive

$$\begin{aligned} C_3 D_{01} \|u_{ty}(t)\|^{\frac{1}{2}} \|u_{\eta\eta}(t)\|^{\frac{1}{2}} &\leq \frac{\varepsilon^2}{2} \|u_{\eta\eta}(t)\|^2 \\ &+ C_3 \varepsilon^{-\frac{2}{3}} (\|u_{ty}(t)\|^2 + \|u_y(t)\|_{H^2}^2 + \|\theta_t(t)\|^2 + \|\theta_{ty}(t)\|^2 + \|\mathbf{b}_t(t)\|^2 + \|\mathbf{b}_{ty}(t)\|^2) \end{aligned} \quad (3.35)$$

and

$$\begin{aligned} C_3 D_{02} \|u_{ty}(t)\|^{\frac{1}{2}} \|u_{\eta\eta}(t)\|^{\frac{1}{2}} &\leq \frac{\varepsilon^2}{2} \|u_{\eta\eta}(t)\|^2 + \varepsilon^2 (\|u_{\eta\eta\eta}(t)\|^2 + \|\mathbf{b}_{\eta\eta}(t)\|^2 + \|\theta_{\eta\eta}(t)\|^2) \\ &+ C_3 \varepsilon^{-6} (\|u_{ty}(t)\|^2 + \|\theta_{ty}(t)\|^2 + \|\mathbf{b}_{ty}(t)\|^2). \end{aligned} \quad (3.36)$$

Thus, we infer from (3.34)-(3.36) that

$$\begin{aligned} D_0 &\leq \varepsilon^2 (\|u_{\eta\eta\eta}(t)\|^2 + \|u_{\eta\eta}(t)\|^2 + \|\mathbf{b}_{\eta\eta}(t)\|^2 + \|\theta_{\eta\eta}(t)\|^2) \\ &+ C_3 \varepsilon^{-6} (\|u_{ty}(t)\|^2 + \|\theta_{ty}(t)\|^2 + \|\mathbf{b}_{ty}(t)\|^2 + \|\theta_t(t)\|^2 + \|u_y(t)\|_{H^2}^2 + \|\mathbf{b}_t(t)\|^2), \end{aligned}$$

which, together with Theorem 1.1, Lemma 2.1, and Lemmas 3.1-3.2, yields

$$\int_0^t D_0 ds \leq \varepsilon^2 \int_0^t (\|u_{\eta\eta\eta}\|^2 + \|u_{\eta\eta}\|^2 + \|\mathbf{b}_{\eta\eta}\|^2 + \|\theta_{\eta\eta}\|^2) (s) ds + C_3 \varepsilon^{-6}. \quad (3.37)$$

Similarly, by Theorem 1.1, Lemma 2.1, and Lemmas 3.1-3.2 and the embedding theorem, we have

$$D_1 \leq (2C_3)^{-1} \|u_{\eta\eta}(t)\|^2 + C_3 (\|u_{ty}(t)\|^2 + \|\mathbf{b}_{ty}(t)\|^2 + \|\theta_t(t)\|_{H^1}^2 + \|u_y(t)\|_{H^1}^2), \quad (3.38)$$

which, combined with (3.33), (3.37)-(3.38), Theorem 1.1, Lemma 2.1, and Lemmas 3.1-3.2, gives that for $\varepsilon \in (0, 1)$ small enough,

$$\|u_{ty}(t)\|^2 + \int_0^t \|u_{ty}(s)\|^2 ds \leq C_3 \varepsilon^{-6} + C_2 \varepsilon^2 \int_0^t (\|b_{ty}\|^2 + \|\theta_{ty}\|^2 + \|u_{tyty}\|^2)(s) ds. \quad (3.39)$$

On the other hand, differentiating (1.2) with respect to x and t , using Theorem 1.1 and Lemmas 3.1-3.2, we have

$$\begin{aligned} \|u_{tyty}(t)\| &\leq C_1 \|u_{ty}(t)\| + C_2 (\|u_{ty}(t)\|_{H^2}^2 + \|\theta_y(t)\|_{H^1}^2 + \|v_y(t)\|_{H^1}^2 \\ &\quad + \|b_y(t)\|_{H^1}^2 + \|\theta_t(t)\|_{H^2}^2 + \|b_t(t)\|_{H^2}^2). \end{aligned} \quad (3.40)$$

Thus, inserting (3.40) into (3.39) implies estimate (3.30).

Analogously, we can obtain estimates (3.31)-(3.32). \square

Lemma 3.4 *Under the assumptions in Theorem 1.2, the following estimates hold for any $t \in [0, T]$,*

$$\begin{aligned} &\|u_{tt}(t)\|^2 + \|u_{ty}(t)\|^2 + \|w_{tt}(t)\|^2 + \|w_{ty}(t)\|^2 + \|b_{tt}(t)\|^2 + \|b_{ty}(t)\|^2 + \|\theta_{tt}(t)\|^2 \\ &\quad + \|\theta_{ty}(t)\|^2 + \int_0^t (\|u_{tty}\|^2 + \|u_{tyty}\|^2 + \|w_{tty}\|^2 + \|w_{tyty}\|^2 + \|b_{tty}\|^2 \\ &\quad + \|b_{tyty}\|^2 + \|\theta_{tty}\|^2 + \|\theta_{tyty}\|^2)(s) ds \leq C_4, \end{aligned} \quad (3.41)$$

$$\|v_{tyty}(t)\|_{H^1}^2 + \|v_{ty}(t)\|_{W^{1,\infty}}^2 + \int_0^t (\|v_{tyty}\|_{H^1}^2 + \|v_{ty}\|_{W^{1,\infty}}^2)(s) ds \leq C_4, \quad (3.42)$$

$$\begin{aligned} &\|u_{tyty}(t)\|_{H^1}^2 + \|u_{ty}(t)\|_{W^{1,\infty}}^2 + \|w_{tyty}(t)\|_{H^1}^2 + \|w_{ty}(t)\|_{W^{1,\infty}}^2 + \|b_{tyty}(t)\|_{H^1}^2 + \|b_{ty}(t)\|_{W^{1,\infty}}^2 \\ &\quad + \|\theta_{tyty}(t)\|_{H^1}^2 + \|\theta_{ty}(t)\|_{W^{1,\infty}}^2 + \|v_{tyty}(t)\|^2 + \|u_{tyty}(t)\|^2 + \|w_{tyty}(t)\|^2 + \|b_{tyty}(t)\|^2 \\ &\quad + \|\theta_{tyty}(t)\|^2 + \int_0^t (\|u_{tt}\|^2 + \|w_{tt}\|^2 + \|b_{tt}\|^2 + \|\theta_{tt}\|^2 + \|u_{ty}\|_{W^{2,\infty}}^2 + \|w_{ty}\|_{W^{2,\infty}}^2 \\ &\quad + \|b_{ty}\|_{W^{2,\infty}}^2 + \|\theta_{ty}\|_{W^{2,\infty}}^2 + \|\theta_{tyty}\|_{H^1}^2 + \|u_{tyty}\|_{H^1}^2 + \|w_{tyty}\|_{H^1}^2 + \|b_{tyty}\|_{H^1}^2 \\ &\quad + \|\theta_{tyty}\|_{H^1}^2 + \|u_{ty}\|_{W^{1,\infty}}^2 + \|w_{ty}\|_{W^{1,\infty}}^2 + \|b_{ty}\|_{W^{1,\infty}}^2 + \|v_{tyty}\|_{H^1}^2)(s) ds \leq C_4, \end{aligned} \quad (3.43)$$

$$\int_0^t (\|u_{tytyty}\|_{H^1}^2 + \|w_{tytyty}\|_{H^1}^2 + \|b_{tytyty}\|_{H^1}^2 + \|\theta_{tytyty}\|_{H^1}^2)(s) ds \leq C_3. \quad (3.44)$$

Proof. Adding up (3.30)-(3.32), picking $\varepsilon \in (0, 1)$ enough small, by Lemmas 3.1-3.3, and Gronwall's inequality, we get

$$\begin{aligned} &\|u_{ty}(t)\|^2 + \|w_{ty}(t)\|^2 + \|b_{ty}(t)\|^2 + \|\theta_{ty}(t)\|^2 + \int_0^t (\|u_{tyty}\|^2 + \|w_{tyty}\|^2 \\ &\quad + \|b_{tyty}\|^2 + \|\theta_{tyty}\|^2)(s) ds \leq C_3 \varepsilon^{-6} + C_2 \varepsilon^2 \int_0^t (\|u_{tty}\|^2 + \|w_{tty}\|^2 \\ &\quad + \|b_{tty}\|^2 + \|\theta_{tty}\|^2 + \|\theta_{tyty}\|^2)(s) ds. \end{aligned} \quad (3.45)$$

Now multiplying (3.24)-(3.26) by ε , ε , and $\varepsilon^{\frac{3}{2}}$, adding the resultant to (3.45), and choosing $\varepsilon \in (0, 1)$ small enough, we obtain

$$\begin{aligned} & \|u_{tt}(t)\|^2 + \|u_{ty}(t)\|^2 + \|w_{tt}(t)\|^2 + \|w_{ty}(t)\|^2 + \|b_{tt}(t)\|^2 + \|b_{ty}(t)\|^2 + \|\theta_{tt}(t)\|^2 \\ & + \|\theta_{ty}(t)\|^2 + \int_0^t (\|u_{ty}\|^2 + \|u_{tyy}\|^2 + \|w_{ty}\|^2 + \|w_{tyy}\|^2 + \|b_{ty}\|^2 \\ & + \|b_{tyy}\|^2 + \|\theta_{ty}\|^2 + \|\theta_{tyy}\|^2)(s) ds \leq C_4 \varepsilon^{-6} + C_2 \varepsilon^2 \int_0^t \|\theta_{ty}(s)\|^2 \|\theta_{yy}(s)\|^2 ds, \end{aligned}$$

which, by Gronwall's inequality, gives the estimate (3.41).

Differentiating (2.12) with respect to y , and using $v_{tyy} = u_{yyy}$, we obtain

$$\lambda \frac{\partial}{\partial t} \left(\frac{v_{yyy}}{v} \right) + \frac{\theta}{v^2} v_{yyy} = E_1(y, t), \quad (3.46)$$

with

$$E_1(y, t) = E_y(y, t) + u_{tyy} + \left(\frac{\theta}{v^2} \right)_y v_{yy} + \lambda \left(\frac{v_y v_{yy}}{v} \right)_t.$$

Obviously, we can infer from Lemmas 3.1-3.3 that

$$\|E_1(t)\| \leq C_2 (\|u_{tyy}(t)\| + \|\theta_y(t)\|_{H^2} + \|u_y(t)\|_{H^2} + \|b_y(t)\|_{H^2} + \|v_y(t)\|_{H^1}) \quad (3.47)$$

leading to

$$\int_0^t \|E_1(s)\|^2 ds \leq C_3. \quad (3.48)$$

Multiplying (3.46) by $\frac{v_{yyy}}{v}$, we get

$$\frac{d}{dt} \left\| \frac{v_{yyy}}{v} \right\|^2 + C_1^{-1} \left\| \frac{v_{yyy}}{v} \right\|^2 \leq C_1 \|E_1(t)\|^2, \quad (3.49)$$

which, combined with (3.48), gives

$$\|v_{yyy}(t)\|^2 + \int_0^t \|v_{yyy}(s)\|^2 ds \leq C_3. \quad (3.50)$$

By (3.4), (3.6), (3.8), (3.10), (3.12), (3.14), (3.17), (3.19), (3.41), (3.50), and Lemmas 3.1-3.3, and using the embedding theorem, we have

$$\begin{aligned} & \|u_{yyy}(t)\|^2 + \|u_{yy}(t)\|_{L^\infty}^2 + \|w_{yyy}(t)\|^2 + \|w_{yy}(t)\|_{L^\infty}^2 + \|b_{yyy}(t)\|^2 + \|b_{yy}(t)\|_{L^\infty}^2 \\ & + \|\theta_{yyy}(t)\|^2 + \|\theta_{yy}(t)\|_{L^\infty}^2 + \int_0^t (\|u_{yy}\|_{W^{1,\infty}}^2 + \|w_{yy}\|_{W^{1,\infty}}^2 + \|b_{yy}\|_{W^{1,\infty}}^2 + \|\theta_{yy}\|_{W^{1,\infty}}^2 \\ & + \|\theta_{yy}\|_{H^1}^2 + \|u_{yy}\|_{H^1}^2 + \|w_{yy}\|_{H^1}^2 + \|b_{yy}\|_{H^1}^2)(s) ds \leq C_3. \end{aligned} \quad (3.51)$$

Differentiating (1.2)-(1.5) with respect to t , using (3.41) and Lemmas 3.1-3.3, we get

$$\|u_{tyy}(t)\| \leq C_1 \|u_{tt}(t)\| + C_1 (\|u_{ty}(t)\| + \|b_{ty}(t)\| + \|\theta_{ty}(t)\|) \leq C_4, \quad (3.52)$$

$$\|w_{tyy}(t)\| \leq C_1 \|w_{tt}(t)\| + C_1 (\|w_{ty}(t)\| + \|b_{ty}(t)\|) \leq C_4, \quad (3.53)$$

$$\|b_{\eta\gamma}(t)\| \leq C_1 \|b_{tt}(t)\| + C_1(\|w_{\eta\gamma}(t)\| + \|b_{\eta\gamma}(t)\|) \leq C_4, \quad (3.54)$$

$$\|\theta_{\eta\gamma}(t)\| \leq C_1 \|\theta_{tt}(t)\| + C_1(\|u_{\eta\gamma}(t)\| + \|w_{\eta\gamma}(t)\| + \|b_{\eta\gamma}(t)\| + \|\theta_{\eta\gamma}(t)\|) \leq C_4, \quad (3.55)$$

which, combined with (3.6), (3.10), (3.14) and (3.19), yields

$$\begin{aligned} & \|u_{\eta\eta\eta}(t)\| + \|w_{\eta\eta\eta}(t)\| + \|b_{\eta\eta\eta}(t)\| + \|\theta_{\eta\eta\eta}(t)\| + \int_0^t (\|u_{\eta\eta\gamma}\|^2 + \|w_{\eta\eta\gamma}\|^2 + \|b_{\eta\eta\gamma}\|^2 \\ & + \|\theta_{\eta\eta\gamma}\|^2 + \|u_{\eta\eta\gamma}\|^2 + \|w_{\eta\eta\gamma}\|^2 + \|b_{\eta\eta\gamma}\|^2 + \|\theta_{\eta\eta\gamma}\|^2)(s)ds \leq C_3. \end{aligned} \quad (3.56)$$

Therefore, it follows from (3.51), (3.56), and the embedding theorem, we obtain

$$\begin{aligned} & \|u_{\eta\eta\eta}(t)\|_{L^\infty} + \|w_{\eta\eta\eta}(t)\|_{L^\infty} + \|b_{\eta\eta\eta}(t)\|_{L^\infty} + \|\theta_{\eta\eta\eta}(t)\|_{L^\infty} \\ & + \int_0^t (\|u_{\eta\eta\gamma}\|_{L^\infty} + \|w_{\eta\eta\gamma}\|_{L^\infty} + \|b_{\eta\eta\gamma}\|_{L^\infty} + \|\theta_{\eta\eta\gamma}\|_{L^\infty})(s)ds \leq C_3. \end{aligned} \quad (3.57)$$

Differentiating (3.46) with respect to y , we obtain

$$\lambda \frac{\partial}{\partial t} \left(\frac{v_{\eta\eta\eta}}{v} \right) + \frac{\theta}{v^2} v_{\eta\eta\eta} = E_2(y, t) \quad (3.58)$$

where $E_2(y, t) = E_{1y}(y, t) + \left(\frac{\theta}{v^2}\right)_y v_{\eta\eta\eta} + \lambda \frac{\partial}{\partial t} \left(\frac{v_y v_{\eta\eta\eta}}{v^2} \right)$.

Using the embedding theorem and Lemmas 3.1-3.3, we can conclude

$$\|E_2(t)\| \leq C_1 \|u_{\eta\eta\eta}(t)\| + C_4(\|\theta_y(t)\|_{H^3} + \|u_y(t)\|_{H^3} + \|b_y(t)\|_{H^3} + \|v_y(t)\|_{H^2}). \quad (3.59)$$

We infer from (3.20)-(3.23) that

$$\int_0^t (\|u_{tt}\|^2 + \|w_{tt}\|^2 + \|b_{tt}\|^2 + \|\theta_{tt}\|^2)(s)ds \leq C_4, \quad (3.60)$$

which, together with Lemma 3.3, gives

$$\int_0^t (\|u_{\eta\eta\eta}\|^2 + \|w_{\eta\eta\eta}\|^2 + \|b_{\eta\eta\eta}\|^2 + \|\theta_{\eta\eta\eta}\|^2)(s)ds \leq C_3. \quad (3.61)$$

Thus, it follows from (3.40), (3.59), (3.61), and Lemmas 3.1-3.3 that

$$\int_0^t \|E_2(s)\|^2 ds \leq C_3. \quad (3.62)$$

Multiplying (3.58) by $\frac{v_{\eta\eta\eta}}{v}$, we get

$$\frac{d}{dt} \left\| \frac{v_{\eta\eta\eta}}{v}(t) \right\|^2 + C_1 \left\| \frac{v_{\eta\eta\eta}}{v}(t) \right\|^2 \leq C_1 \|E_2(t)\|^2, \quad (3.63)$$

whence by (3.62),

$$\|v_{\eta\eta\eta}(t)\|^2 + \int_0^t \|v_{\eta\eta\eta}(s)\|^2 ds \leq C_3. \quad (3.64)$$

Differentiating (1.2) with respect to y there times, using Lemmas 3.1-3.3 and Poincaré's inequality, we have

$$\|u_{yyyy}(t)\| \leq C_3 \|u_{yy}(t)\| + C_3(\|v_y(t)\|_{H^3} + \|u_y(t)\|_{H^3} + \|\theta_y(t)\|_{H^3} + \|b_y(t)\|_{H^3}). \quad (3.65)$$

Thus, we conclude from (1.2), (3.56), (3.61), (3.64), and (3.65) that

$$\int_0^t (\|u_{yyyy}\|^2 + \|v_{yyy}\|_{H^1}^2)(s) ds \leq C_3. \quad (3.66)$$

Similarly, we can deduce from (1.3)-(1.5) that

$$\int_0^t (\|b_{yyyy}\|^2 + \|w_{yyy}\|^2 + \|\theta_{yyy}\|^2)(s) ds \leq C_4. \quad (3.67)$$

which, along with (3.51) and (3.66), gives

$$\int_0^t (\|u_{yy}\|_{W^{2,\infty}}^2 + \|w_{yy}\|_{W^{2,\infty}}^2 + \|b_{yy}\|_{W^{2,\infty}}^2 + \|\theta_{yy}\|_{W^{2,\infty}}^2)(s) ds \leq C_3. \quad (3.68)$$

Finally, using (1.1), (3.50)-(3.56), (3.64), (3.66)-(3.68), and Sobolev's interpolation inequality, we can get the desired estimates (3.42)-(3.44).

Proof of Theorem 1.2. By Lemma 2.1, Lemmas 3.1-3.4, and Theorem 1.1, we complete the proof of Theorem 1.2.

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Authors' contributions

All authors contributed to each part of this work equally.

Competing interests

The authors declare that they have no competing interests.

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